Symmetry analysis and numerical simulation of mode characteristics for equilateral-polygonal optical microresonators

Yue-De Yang and Yong-Zhen Huang
State Key Laboratory on Integrated Optoelectronics, Institute of Semiconductors, Chinese Academy of Sciences, P.O. Box 912, Beijing 100083, China
(Received 9 February 2007; published 22 August 2007)

Mode characteristics for two-dimensional equilateral-polygonal microresonators are investigated based on symmetry analysis and finite-difference time-domain numerical simulation. The symmetries of the resonators can be described by the point group $C_{Nv}$, accordingly, the confined modes in these resonators can be classified into irreducible representations of the point group $C_{Nv}$. Compared with circular resonators, the modes in equilateral-polygonal resonators have different characteristics due to the break of symmetries, such as the split of double-degenerate modes, high field intensity in the center region, and anomalous traveling-wave modes, which should be considered in the designs of the polygonal resonator microlasers or optical add-drop filters.

DOI: 10.1103/PhysRevA.76.023822 PACS number(s): 42.60.Da, 02.20.—a, 42.55.Sa

I. INTRODUCTION

Semiconductor microresonators [1] have attracted great attention because of their potential applications in light sources and optical add-drop filters for wavelength-division multiplexing. As typical representatives, circular resonators [2–6] and deformed circular resonators [7–9] are successfully used in the fabrication of microlasers and add-drop filters. High quality factor ($Q$ factor) optical modes are confined by the total internal reflection in such microresonators. Recently, polygonal microresonators have also been applied in the microcavity lasers and add-drop filters, such as triangular [10–12], square [13–16], hexagonal [17,18], and octagonal microresonators [19]. Ray optics and paraxial approximation are used to study the mode characteristics for such resonators [10,13,19], but they fail to provide reliable results for the resonators with size comparable to the mode wavelength. The boundary element method is also used to compute resonances in 2D dielectric resonators [20]. We have derived the analyzed mode distributions and wave-lengths for the equilateral triangle and square resonators [11,12,14].

In this article, we investigate the mode characteristics for wavelength scale equilateral-polygonal microresonators by the symmetry analysis and finite-difference time-domain (FDTD) numerical simulation. Based on the symmetries of an equilateral-polygonal microresonator, the confined modes in the microresonator can be classified into irreducible representations of the point group $C_{Nv}$ [21], where $N$ is the side number of the equilateral polygonal resonator. Different from double-degenerate whispering-gallery (WG) modes in the circular microresonator, the confined modes of one-dimensional (1D) representations in the equilateral-polygons are nondegenerate modes due to the break of the symmetries. However, the other modes of 2D representations are still double-degenerate modes. Furthermore, the mode distributions include several angular components with the wave numbers of congruent modulo $N$, because the field distributions with such wave numbers can form the same irreducible representation and cannot be divided by the symmetries. In fact, the scattering of the periodic sidewall induces multiple angular wave components and results in the changes of mode characteristics, such as high field intensity in the center region, which must be considered in the pedestal structure polygonal cavities [22], and the traveling-wave modes exhibit some mode characteristics of standing-wave modes.

II. SYMMETRY ANALYSIS FOR EQUILATERAL-POLYGONAL MICRORESONATORS

In this section, we analyze the mode symmetry characteristics for equilateral-polygonal microresonators based on group theory. An illustration of a 2D equilateral-polygon is shown in Fig. 1. The center of the cavity $O$ is the coordinate origin, $n_1$ and $n_2$ are the refractive indices of the inner and external regions, $R_1$ and $R_2$ are the maximum and minimum radii of the resonators, respectively. The symmetries of the equilateral-polygonal microresonator can be described by the point group $C_{Nv}$. $\sigma_1$ and $\sigma_2$ are mirror reflection elements and $C_N$ is the rotational subgroup of $C_{Nv}$. The $z$-direction magnetic field $H_z$ of the TE modes and electric field $E_z$ of the TM modes in the equilateral-polygonal microresonator can be expressed as $F_z(r, \varphi)$, where $r$ and $\varphi$ are the radius and angle in the polar coordinates, respectively. $F_z(r, \varphi)$ satisfies the 2D Helmholtz equation

![FIG. 1. An illustration of a 2D equilateral-polygonal microresonator. The operators of the point group $C_{Nv}$ as well as the coordinates are indicated](image-url)
TABLE I. Character table of the point group $C_{N \sigma}$ for even $N$.

<table>
<thead>
<tr>
<th>$C_{N \sigma}$</th>
<th>$E$</th>
<th>$C_{N \sigma}^m$</th>
<th>$N/2 \sigma_1$</th>
<th>$N/2 \sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$A_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$B_1$</td>
<td>1</td>
<td>(-1)$^m$</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$B_2$</td>
<td>1</td>
<td>(-1)$^m$</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

$$
\left( \frac{\partial^2}{\partial r^2} + 1 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) F_z(r, \varphi) + [kn(r, \varphi)]^2 F_z(r, \varphi) = 0, 
$$

(1)

where $k$ is wave vector in vacuum, $n$ is the refractive index and equals to $n_1$ and $n_2$ in the inner and external regions of the microresonator, respectively. We expand the field distribution by $\exp(iuv)$ as

$$
F_z(r, \varphi) = \sum_{v=0}^{\infty} f_v(r) \exp(iuv) = \sum_{v=0}^{\infty} g_{1,v}(r) \cos(v\varphi) + \sum_{v=1}^{\infty} g_{2,v}(r) \sin(v\varphi),
$$

(2)

where $v$ is the angular wave number, $g_{1,v}(r)=f_v(r)+f_{-v}(r)$ ($v \neq 0$), $g_{1,0}(r)=f_0(r)$, $g_{2,v}(r)=i[f_v(r)-f_{-v}(r)]$, and $f$ and $g$ are the field distributions in the $r$ direction for traveling and standing waves, respectively.

In 2D equilateral-polygonal microresonator, the modes can be classified into the irreducible representations of the point group $C_{N \sigma}$, and the modes in different classes can be divided by symmetries. When $N$ is even, the characters of point group $C_{N \sigma}$ are listed in Table I, where $1 \leq m \leq N-1$ and $1 \leq n \leq N/2 - 1$. Based on the group operations and the character table, we find that the mode field distributions

$$
A_{1z}(r, \varphi) = \sum_{l=0}^{\infty} g_{1,IN}(r) \cos(lN\varphi),
$$

(3)

$$
A_{2z}(r, \varphi) = \sum_{l=0}^{\infty} g_{2,IN}(r) \sin(lN\varphi),
$$

(4)

$$
B_{1z}(r, \varphi) = \sum_{l=0}^{\infty} g_{1,(l+1/2)N}(r) \cos[(l+1/2)N\varphi],
$$

(5)

$$
B_{2z}(r, \varphi) = \sum_{l=0}^{\infty} g_{2,(l+1/2)N}(r) \sin[(l+1/2)N\varphi],
$$

(6)

form the $A_1$, $A_2$, $B_1$, and $B_2$ representations of point group $C_{N \sigma}$, respectively. Furthermore, the mode field distributions

$$
G_{n_z}(r, \varphi) = \sum_{l=\infty}^{\infty} f_{ln+n}(r) \exp[i(lN+n)\varphi] 
$$

and $G_{n_z}^*(r, \varphi)$ form the $E_n$ representations. When $N$ is odd, the characters of point group $C_{N \sigma}$ are listed in Table II, where $1 \leq m \leq N-1$ and $1 \leq n \leq (N-1)/2$. Similar to even $N$, the mode field distributions $A_{1z}(r, \varphi)$ and $A_{2z}(r, \varphi)$ form the $A_1$ and $A_2$ representations of point group $C_{N \sigma}$, respectively. Furthermore, the mode field distributions $G_{n_z}(r, \varphi)$ and $G_{n_z}^*(r, \varphi)$ form the $E_n$ representations. The modes with field distributions of $A_{1z}(r, \varphi)$, $A_{2z}(r, \varphi)$, $B_{1z}(r, \varphi)$, or $B_{2z}(r, \varphi)$ are nondegenerate modes, and those modes with field distributions of $G_{n_z}(r, \varphi)$ and $G_{n_z}^*(r, \varphi)$ are double-degenerate modes.

The double-degenerate modes with traveling-wave distribution (7) can also be expressed as standing-wave modes $G'_{n_z}$ with field distributions

$$
G'_{1z}(r, \varphi) = [G_{n_z}(r, \varphi) + G_{n_z}^*(r, \varphi)]/\sqrt{2},
$$

(8)

$$
G'_{2z}(r, \varphi) = [G_{n_z}(r, \varphi) - G_{n_z}^*(r, \varphi)]/i\sqrt{2},
$$

(9)

which still form the $E_n$ representations. In fact, the traveling- and standing-wave modes are different representations of the double-degenerate modes, which are corresponding to the eigenstates of group operations $C_N^1$ and $\sigma_1$, respectively. If only if the modes can be expressed as the eigenstates of both $C_N^1$ and $\sigma_1$, the modes are nondegenerate modes, such as the modes in Eqs. (3)–(6).

Circular microresonators are special equilateral-polygonal microresonators with side number $N \rightarrow \infty$, so all modes in such microresonators are double-degenerate modes except angular wave number $v=0$. The double-degenerate modes should be split when the corresponding angular wave number $v=LN$ for odd $N$ and $v=LN$ or $v=(l+1/2)N$ for even $N$, which is observed in the triangular and square microresonators [12,24]. Because the field distributions with the angular components of wave numbers that are congruent modulo $N$ can form the same irreducible representations and cannot be divided by the symmetries. The phenomena can also be ascribed to the scattering due to the asymmetrical distribution of the refractive indices in the $R_3<r<R_1$ region. In fact, the symmetry analysis suits the cavities with the symmetries of the point group $C_{N \sigma}$, such as 2D photonic crystal cavities [23], deformed polygonal microresonators [24], and microresonators with periodic sidewall surface roughness [25].

### III. NUMERICAL SIMULATIONS

To examine the symmetry characteristics of mode distributions in the polygonal microresonators, we calculate the
mode field distributions and $Q$ factors by the FDTD simulation. The FDTD techniques [26] are used to simulate the mode characteristics for the wavelength scale equilateral-polygonal microresonator as shown in Fig. 1. In the simulation, the spatial cell size is 10 nm, and the time step $\Delta t$ is Courant limit. The side length $a$ is taken to be 3, 2, 1, and 0.8 $\mu$m for the equilateral triangular, square, hexagonal, and octagonal microresonators, respectively. The refractive indices $n_1$ and $n_2$ are equal to 3.2 and 1, respectively. An exciting source with a cosine impulse modulated by a Gaussian function $P(x_0, y_0, t) = \exp[-(t-t_0)^2/r^2_0] \cos(2\pi f_0 t)$ is added to one component of the electromagnetic fields at a point $(x_0, y_0)$ inside the microresonator, where $t_0$, $f_0$, and $f_0$ are the time of the pulse center, the pulse half width and the center frequency of the pulse, respectively. The perfect matched layer (PML) absorbing boundary condition is used as the boundary to terminate the FDTD computation window [27]. The Padé approximation with Baker's algorithm [28] is used to transform the FDTD output from the time domain to the frequency domain, and then the mode frequencies and $Q$ factors are calculated from the obtained intensity spectrum. Finally, a long optical pulse with a very narrow bandwidth with center frequency as a mode frequency is used to excite only one mode and calculate the corresponding mode field distribution by the FDTD simulation.

The TM modes are used as examples in the FDTD simulations. The WG modes $\text{WGM}_{v,m}$ in circular microresonators are double-degenerate modes, where $v$ and $m$ are angular and radial mode numbers, respectively. These modes transform to corresponding confined modes when circular microresonators deform to equilateral-polygonal microresonators [24], then we mark the confined modes in the equilateral-polygonal microresonators as $\text{TM}_{v,m}$, with angular and radial mode numbers $v$ and $m$. In the equilateral triangle [12] and square [14] microresonators, the modes were marked as $\text{TM}(m_t, m_r)$, where $m_t$ and $m_r$ are longitudinal and transverse mode numbers, respectively. $\text{TM}_{v,m}$ and $\text{TM}(m_t, m_r)$ are the same mode when the mode numbers $v$, $m_t$, and $m_r$ satisfy $v=m_t=N(N=3, 4)$ and $n_r=m_r+1$.

First, we simulate the nondegenerate modes with standing-wave distribution, by applying symmetric and antisymmetric sources relative to $\sigma_1$ to excite the corresponding modes. The field distributions of $E_z$ are plotted in Fig. 2 for (a) $\text{TM}_{6,1}$, (b) $\text{TM}_{8,1}$, (c) $\text{TM}_{6,1}$, and (d) $\text{TM}_{8,1}$ modes in the triangle, square, hexagonal, and octagonal microresonators, respectively. The left and right field distributions are $A_1$ and $A_2$ representations, and the mode forming $A_2$ representations also has higher mode frequencies and $Q$ factors. $\text{TM}_{9,1}$ vertical waveguiding in such microresonators. Then different field distributions in the center region can result in a mode selection [22]. The modes with field distributions of $A_1$ representations will be suppressed in the microresonators with a pedestal, so the polygonal resonators would have advantage to realize real single mode operation.

We also simulate $\text{TM}_{6,2}$ and $\text{TM}_{9,1}$ modes for the hexagonal microresonator. The field patterns of $\text{TM}_{6,2}$ and $\text{TM}_{9,1}$ modes are shown in Fig. 3. The mode frequencies and $Q$ factors for $\text{TM}_{6,2}$, and $\text{TM}_{9,1}$ modes are listed in Table IV. Similar to the fundamental mode $\text{TM}_{6,1}$, the second order mode $\text{TM}_{6,2}$ splits into two standing-wave modes forming $A_1$ and $A_2$ representations, and the mode forming $A_2$ representations also has higher mode frequencies and $Q$ factors. $\text{TM}_{9,1}$

<table>
<thead>
<tr>
<th>$A_1$ representations</th>
<th>$A_2$ representations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$(THz)</td>
<td>$Q$ factors</td>
</tr>
<tr>
<td>triangular</td>
<td>194.79 $\times 10^3$</td>
</tr>
<tr>
<td>square</td>
<td>148.79 $\times 2$</td>
</tr>
<tr>
<td>hexagonal</td>
<td>142.42 $\times 3$</td>
</tr>
<tr>
<td>octagonal</td>
<td>166.26 $\times 4$</td>
</tr>
</tbody>
</table>

TABLE III. Mode frequencies and $Q$ factors for $\text{TM}_{9,1}$, $\text{TM}_{8,1}$, $\text{TM}_{6,1}$, and $\text{TM}_{8,1}$ modes in the triangular, square, hexagonal, and octagonal microresonators, respectively.
FIG. 3. The field patterns of (a) \( A_1 \) and (b) \( A_2 \) representations of TM_{6,2} mode, and (c) \( B_1 \) and (d) \( B_2 \) representations of TM_{9,1} mode in the 2D hexagonal microresonator with side length \( a=1 \ \mu m \) obtained by FDTD technique.

mode is corresponding to angular wave number \( v=(l+1/2)N \), then the mode splits into two standing-wave modes forming \( B_1 \) and \( B_2 \) representations.

Then, we consider the double-degenerate modes, which can be divided into two traveling-wave modes, in the hexagonal microresonator. The traveling-wave mode field distributions \( G_n{c}(r,\varphi) \) and \( G_{n}\prime{c}(r,\varphi) \) can be calculated from the normalized field distributions of the standing-wave modes \( G_{1,n}(r,\varphi) \) and \( G_{2,n}(r,\varphi) \) obtained by the FDTD simulation as

\[
G_n{c}(r,\varphi) = G_{1,n}(r,\varphi) + iG_{2,n}(r,\varphi).
\]

Similar to the simulations for nondegenerate modes, the exciting sources with the same symmetries as the eigenstates of the mirror reflection element \( \sigma_1 \) are used to excite the modes with the eigenvalue 1 and \(-1\), respectively. The two eigenstates have the same mode frequency and \( Q \) factor. The obtained mode field distribution for TM_{5,1} and TM_{7,1} modes over a half of a period \( T \) with a time interval of \( T/8 \) are plotted in Figs. 4(a) and 4(b) for the hexagonal microresonator with side length of \( 1 \ \mu m \). Both the peripheral field of TM_{5,1} and TM_{7,1} modes and the inner field of TM_{7,1} mode propagate anticlockwise, however, the inner field of TM_{5,1} mode propagates clockwise. Expanding the traveling-wave field distributions of TM_{5,1} and TM_{7,1} modes by \( \exp(i\varphi) \),

\[
E_c(r,\varphi) = \sum_{n=\infty}^\infty f_n(r)\exp(i\varphi),
\]

we find that TM_{4,1} mode only has two obvious angular components at \( v=7 \) and \( 1 \). When the time dependence \( \exp(-i\omega t) \) is used, \( v>0 \) and \( v<0 \) are corresponding to the anticlockwise and clockwise propagating components, respectively. Anticlockwise and clockwise propagating components exist in one traveling-wave mode would lead some standing-wave characteristics to the modes. Figure 5 shows the field distributions of the main angular wave numbers in \( r \) direction for TM_{5,1} and TM_{7,1} modes, which can be fitted by the corresponding Bessel functions very well when \( r<R_2 \), i.e.,

\[
f_n(r) = a_n J_n(nkr), \quad r < R_2,
\]

where \( J_n \) are the Bessel functions.

![FIG. 4](image_url) The field patterns of the traveling-wave modes vary over a half of a period \( T \) with a time interval of \( T/8 \) for (a) TM_{5,1} and (b) TM_{7,1} modes in the hexagonal microresonator with side length \( a=1 \ \mu m \).

![FIG. 5](image_url) The field distributions \( f_n(r) \) for (a) TM_{5,1}, and (b) TM_{7,1} modes.

<table>
<thead>
<tr>
<th>TABLE IV.</th>
<th>Mode frequencies and ( Q ) factors for TM_{6,2}, and TM_{9,1} modes in the hexagonal microresonators, respectively.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1(B_1) ) representations</td>
<td>( A_2(B_2) ) representations</td>
</tr>
<tr>
<td>( f ) (THz)</td>
<td>( Q ) factors</td>
</tr>
<tr>
<td>WGM_{6,2}</td>
<td>197.14</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The mode frequencies and $Q$ factors of the TM$_{m,1}$ modes in the hexagonal microresonator are plotted in Fig. 6 as functions of angular mode number $v$. $C_1$ and $C_2$ are the eigenstates of $\sigma_1$ with eigenvalue 1 and $-1$, respectively. $C_1$ is $G'_{1,1}; A_{1,2}$, $G'_{1,1}; C_{1,2}$, $B_{1,2}$, $G'_{1,1}$ modes, and $C_2$ is $G'_{2,1}$, $A_{2,2}$, $G'_{2,2}$, $C_{2,2}$, $B_{2,2}$, $G'_{2,2}$, and $G'_{2,2}$ modes as the angular mode number $v=5$, 6, 7, 8, 9, 10, and 11, respectively. We find that $G'_{1,1}$ and $G'_{2,2}$ have the same mode frequencies and $Q$ factors, because they are double-degenerate modes. The modes of $A_2$ ($B_2$) representations have larger mode frequencies than the corresponding modes of $A_1$ ($B_1$) representations. In a microdisk resonator with a radius of 1 $\mu$m and the refractive index 3.2, the $Q$ factors of the fundamental WG modes are $1.02 \times 10^3$, $4.42 \times 10^3$, $1.99 \times 10^4$, $9.18 \times 10^4$, $4.32 \times 10^5$, $2.06 \times 10^6$, and $9.99 \times 10^6$ for $v=5$, 6, 7, 8, 9, 10, and 11, respectively. The $Q=4.42 \times 10^3$ for WGM$_{6,1}$ in the microdisk is only a little bigger than $4.3 \times 10^3$ and $3.4 \times 10^3$ of the two nondegenerate TM$_{6,1}$ modes in hexagonal resonators. The $Q$ factors of the other TM modes with $(v \neq 6)$ in Fig. 6 are much smaller than those of the WGMs in the microdisk. The angular mode number equals to the number of sides of the hexagonal resonator may result in a large $Q$ factor for TM$_{6,1}$ due to the same symmetry. It should be noted that the decrease of $Q$ factors with the increase of the angular mode number in Fig. 6 is not contrary to the results of Ref. [18]. For the wave-length scale microresonators considered in this article, the corresponding Re($kR$) is less than the minimum value 20 in the figures of Ref. [18].

Finally, we calculate the traveling-wave field distribution TM$_{11,1}$ mode in the triangular, square, hexagonal, and octagonal microresonators, and expand the field distributions by exp($iv\varphi$). Many peaks with $v=11 \pm N$ appear, positive and negative values of $v$ mean the anticlockwise and clockwise propagating waves in the one mode. When $N=3$ and 4, no main angular components exist, so the field distributions in the triangular and square microresonators are very different from those in circular microresonators. When $N=6$ and 8, a main angular component at $v=11$ is observed, and the mode characteristics are more similar to those in circular microresonators. Through the simulation, we find that mode field distributions in equilateral-polygonal microresonators are the sum of wave functions with multiple values of $v$. This universal phenomenon is induced by the scattering of the vertices of the polygon. The periodic cavity shape would result in an angular wave number selectivity, which can be obtained by symmetry analysis.

IV. CONCLUSION

In summary, we analyze the mode characteristics for the 2D equilateral-polygonal microresonators based on group theory, and classify the modes into the irreducible representations of the point group $C_{6v}$. The modes are divided into double-degenerate and nondegenerate modes based on their mode index. The nondegenerate modes are corresponding to mode index $v=\pm N$ for odd $N$ and $v=\pm N$ or $v=(l+1)/2N$ for even $N$, respectively. Through the FDTD simulations, we find that the two nondegenerate modes have different mode frequencies and $Q$ factors, and single mode field distribution has many angular components, which agree well with the symmetry analysis.

ACKNOWLEDGMENTS

This work was supported by the National Nature Science Foundation of China under Grant No. 60225011 and the Major State Basic Research Program under Grant No. 2006CB302804.