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# An explicit finite element-finite difference method for analyzing the effect of visco-elastic local topography on the earthquake motion*

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**Abstract**

An explicit finite element-finite difference method for analyzing the effects of two-dimensional visco-elastic local topography on earthquake ground motion is proposed in this paper. In the method, at first, the finite element discrete model is formed by using the artificial boundary and finite element method, and the dynamic equations of local nodes in the discrete model are obtained according to the theory of the special finite element method similar to the finite difference method, and then the explicit step-by-step integration formulas are presented by using the explicit difference method for solving the visco-elastic dynamic equation and Generalized Multi-transmitting Boundary. The method has the advantages of saving computing time and computer memory space, and it is suitable for any case of topography and has high computing accuracy and good computing stability.

**Key words: visco-elastic, seismic response, finite difference method**, explicit finite element, artificial boundary

## 1 Introduction

The analyses of the effects of local site conditions on earthquake ground motion play important roles in the aseismic design of constructions. The effects of local site condition on earthquake ground motion include the topographic effects of local site and the inhomogeneous and nonlinear effects of medium. If only considering the topographic effects of local site, we consider the local site condition by linear and homogeneous half-space model, and if considering the inhomogeneous effects of local site, especially considering the nonlinear effects, the analytic model of local site is complex and there are different models for different cases of local site. For the problem of the topographic effects, because the medium is considered as a linear and homogeneous one, there are little restricted conditions for selecting the analysis methods, and many kinds of semi-analytic methods and numerical methods are applicable, such as boundary element method (Du *et al.*, 1993), semi-analytic boundary element method (Zhao *et al.*, 1993), finite element method, finite difference method, finite element-finite difference method (Liao, 1984). For the problem of the regular topographic site, for example, the topographic form of concave and convex spherical surface, analytic method is effective (Liu, 1989). But for the problem considering the inhomogeneous and nonlinear effects of site, the applicable method must be numerical method, such as element method, finite difference method and finite element-finite difference method. The previous

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researches and the practical experiences in engineering show that explicit step-by-step integration methods have an advantage over other ones not only in saving computing time but also in saving computer memory space for the problem of complex local site, especially, if considering the non-linear properties of medium. Zhen-Peng LIAO and his colleagues have devoted to the research work and constructed an explicit finite element-finite difference method (Liao, 1984). The method is based on finite element method in space and center finite difference method in time. Because center finite difference method in time is no longer an explicit method but an implicit one for solving the dynamic equations of visco-elastic multi-lumped-mass system (the damping matrix is not a diagonal matrix), for this case, an approximate measurement (Yang *et al.*, 1991) has to be introduced in the method to make the method explicit, but it results in the reduction of computing accuracy.

In this paper, an explicit high accuracy step-by-step integration formula of the explicit finite element-finite difference method is presented. Applied in analyzing the visco-elasto-plastic earthquake response of site, the new formula gets rid of the shortcomings of the previous formulas, such as not an explicit formula or low computing accuracy. In order to clearly express the basic idea of our explicit finite element-finite difference method, the visco-elastic topographic model of site is used as the analytic model in the paper.

## 2 The mechanical model of site and the dynamic equations

The two-dimensional visco-elastic topographic model is our analytic model. The analytic model with artificial boundaries is shown in Figure 1.

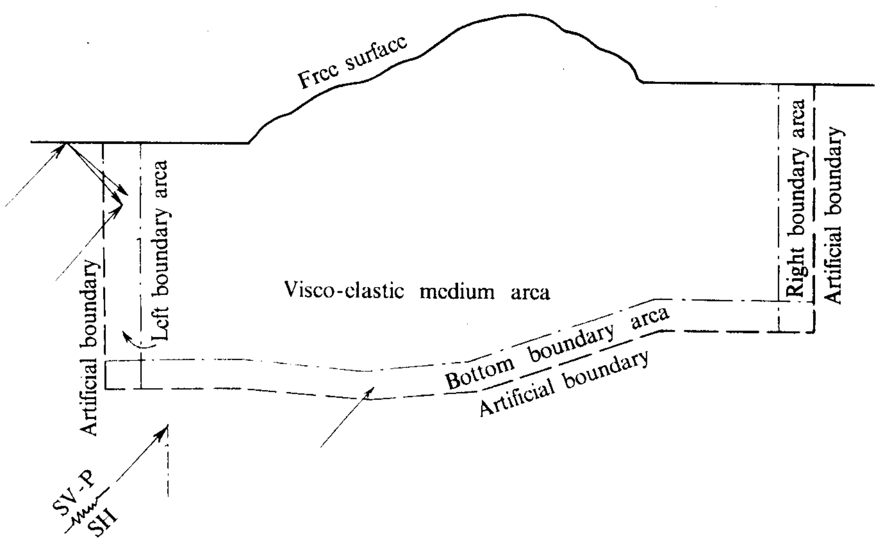



Figure 1 Sketch map of two-dimensional topographic model of site

The finite element discrete model in space about the computing area within the artificial boundaries can be obtained by finite element technique. Assuming that the medium in the computing area is visco-elastic isotropic body, the corresponding dynamic equations of the nodes in the finite element discrete model are

$$[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = \{R(t)\} \tag{1}$$

In Eq. (1), $[M]$, $[C]$ and $[K]$ are the assembly mass matrix, damping matrix and stiffness of the finite element system, respectively; $\{\ddot{u}(t)\}$, $\{\dot{u}(t)\}$ and $\{u(t)\}$ are the acceleration vector, velocity vector and displacement vector of the nodal motions, respectively; $\{R(t)\}$ is the nodal loading vector.

If the damping effects of the medium are considered by the Rayleigh damping theory, we have

$$[C] = \alpha[M] + \beta[K] \tag{2}$$

in which, $\alpha$, $\beta$ are the scaling factors which control the degrees of the damping effects.

For the finite element model with numbers of elements, excessive computer memory space and excessive computing time are needed to solve the dynamic equations if the matrix $[M]$, $[C]$ and $[K]$ in Eq. (1) are formed directly by the traditional finite element method. Therefore, the special finite element method similar to finite difference method is introduced in the paper to set up the dynamic equations of nodal motions for the local nodal system in whole computing model (Liao, 1984).

**2.1 The stiffness matrix, mass matrix and nodal loading vector of element**

For every quadrilateral planar element with four nodes, as shown in Figure 2, following basic formulas are true

The stiffness matrix of element

$$[K_{ij}^e] = \iint_{se} [B]^T[D][B]t \, dx dy \tag{3}$$

The consistent mass matrix of element

$$[M_{ij}^e] = \iint_{se} [N]^T \rho [N] t \, dx dy \tag{4}$$

The equivalent nodal loading vector of element

$$\{R_i^e\} = \iint_{se} [N]^T g t dx dy + \int_{Le} [N]^T \tau t \, dl \tag{5}$$

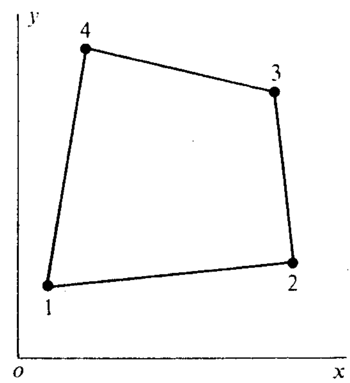



Figure 2 Sketch map of quadrilateral element with four nodes

In above formulas, $\rho$ is the density of the medium in element, assume that it is constant in element; $[N]$ is the shape function matrix of element; $[B]$ is the geometric matrix of element; $t$ is the thickness of element, it is constant in element; $[D]$ is the material matrix, and is given by for SH wave problem

$$[D] = \rho \begin{bmatrix} C_s^2 & 0 \\ 0 & C_s^2 \end{bmatrix} \tag{6}$$

for SV-P wave problem

$$[D] = \rho \begin{bmatrix} C_P^2 & C_P^2 - 2C_s^2 & 0 \\ C_P^2 - 2C_s^2 & C_P^2 & 0 \\ 0 & 0 & C_s^2 \end{bmatrix} \tag{7}$$

in which, $C_S$, $C_P$ are the velocities of S wave and P wave of the medium in the element.

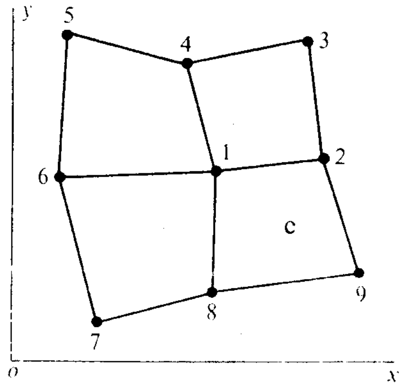



Figure 3 Sketch map of the element connections in local nodal system

### 2.2 The dynamic equations of local nodal motions

For the local nodal system as shown in Figure 3, a kind of lumped mass finite element method similar to finite difference method (Liao, 1984) is used to set up the dynamic equations of node No. 1. In the method, the shape function matrix and the lumped mass matrix are given based on following basic assumptions:

1) The changes of the strains in element are higher order small values comparing with the strain values, the changes can be neglected;

2) The changes of the inertia force in element are higher order small values comparing with the inertial values, the changes can be neglected.

In Figure 3, $L$ is the total number of nodes in the local nodal system, and the ordinal number of nodes is described by $l$, $l=1, 2, \cdots, L$. The displacement vector of the nodal motions in the nodal system is described by $\boldsymbol{u}$, and

$$[\boldsymbol{u}]^T = [\boldsymbol{u}_1\ \boldsymbol{u}_2 \cdots \boldsymbol{u}_L] \tag{8}$$

in Eq. (8),
for SH wave problem

$$\boldsymbol{u}_i = [\boldsymbol{u}_{ix}] \tag{9a}$$

for SV-P wave problem

$$\boldsymbol{u}_i = [\boldsymbol{u}_{ix}\boldsymbol{u}_{iy}] \tag{9b}$$

In the nodal system, the relation between the nodal ordinal number 1 and the nodal ordinal number $j$ in one of the elements in the system is

$$l = l(j) \tag{10}$$

Correspondingly, the following relation is true

$$\boldsymbol{u}_l = \boldsymbol{u}_{l(j)} = \boldsymbol{u}_j^e \tag{11}$$

and especially, stipulate that the node with ordinal number 1 in local nodal system also takes ordinal number 1 in each element, that is

$$\boldsymbol{u}_1 = \boldsymbol{u}_1^e \tag{12}$$

Combining above assumption (1) with the formulas of the stiffness matrix and mass matrix of element given above, for the node No. 1, we have

$$\sum_e [\sum_{j=1}^{4} M_{1j}^e \ddot{u}_j^e + \sum_{j=1}^{4} (\alpha M_{1j}^e + \beta K_{1j}^e)\dot{u}_j^e + \sum_{j=1}^{4} K_{1j}^e u_j^e - R_1^e] = 0 \tag{13}$$

According to the above assumption (2), we have

$$\ddot{u}_{(j)}^e = \ddot{u}_1 \tag{14}$$

From the Eqs. (13) and (14), the following formula can be obtained

$$M_1\ddot{u}_1 + \sum_{l=1}^{L} (\alpha M_{1l} + \beta K_{1l})\dot{u}_l + \sum_{l=1}^{L} K_{1l} u_l - R_1 = 0 \tag{15}$$

in which

$$M_1 = \sum_e \sum_{j=1}^{4} M_{1j}^e \tag{16}$$

$$\sum_{l=1}^{L} M_{1l} = \sum_e \sum_{j=1}^{4} M_{1j}^e \tag{17}$$

$$\sum_{l=1}^{L} K_{11} = \sum_e \sum_{j=1}^{4} K_{1j}^e \tag{18}$$

$$R_1 = \sum_e R_1^e \tag{19}$$

The Eq. (15) is the dynamic equation of node No. 1 in the nodal system. On the analogy of this, the relations of the response quantities between each node and the related nodes can be obtained. If there are some other elements with shape other than quadrilateral element, such as triangle element, in the finite element system, the dynamic equations of the local nodal system can also be obtained by similar idea introduced above.

## 3 The explicit difference solution of the dynamic equations of local nodes

In order to get an explicit high accuracy step-by-step integration formula to analyze the earthquake response of site, the explicit finite difference method for solving the visco-elasto-plastic dynamic equations proposed recently by the authors of this paper (Li *et al.*, 1992) is applied in this paper, the computing accuracy of this method is not lower than two-order.

The explicit finite difference solution of the Eq. (1) is

$$\left.\begin{aligned}
\{u^{P+1}\} &= \frac{1}{2}\Delta t^2[M]^{-1}\{R^P\} + ([I] - \frac{1}{2}\Delta t^2[M]^{-1}[K])\{u^P\} + (\Delta t[I] \\
&\quad - \frac{1}{2}\Delta t^2[M]^{-1}[C])\{\dot{u}^P\} \\
\{\dot{u}^{P+1}\} &= \frac{1}{2}\Delta t[M]^{-1}(\{R^{P+1}\} + \{R^P\}) + \{\dot{u}^P\} - (\frac{1}{2}\Delta t[M]^{-1}[K] + [M]^{-1}[C])\{u^{P+1}\} \\
&\quad - (\frac{1}{2}\Delta t[M]^{-1}[K] - [M]^{-1}[C])\{u^P\} \\
\{\ddot{u}_1^{P+1}\} &= -\{\ddot{u}_1^P\} + \frac{2}{\Delta t}(\{\dot{u}_1^{P+1}\} - \{\dot{u}_1^P\})
\end{aligned}\right\} \tag{20}$$

in which, $P$ means the computing time $P\Delta t$, $[\boldsymbol{I}]$ is the a unit matrix of which order is the same as that of matrix $[\boldsymbol{K}]$.

The explict finite difference solution of the Eq. (15) is

$$\left.\begin{aligned}
\boldsymbol{u}_1^{P+1} &= \frac{\Delta t^2}{2\boldsymbol{M}_1}\boldsymbol{P}_1 + \boldsymbol{u}_1^P + \Delta t\dot{\boldsymbol{u}}_1^P - \frac{\Delta t^2}{2\boldsymbol{M}_1}\sum_{l=1}^{L}[\boldsymbol{K}_{1l}\boldsymbol{u}_l^P + (\alpha\boldsymbol{M}_{1l} + \beta\boldsymbol{K}_{1l})\dot{\boldsymbol{u}}_l^P] \\
\dot{\boldsymbol{u}}_1^{P+1} &= \frac{\Delta t}{2\boldsymbol{M}_1}(\boldsymbol{R}_1^{P+1} + \boldsymbol{R}_1^P) + \dot{\boldsymbol{u}}_1^P - \frac{1}{2\boldsymbol{M}_1}\sum_{l=1}^{L}[\boldsymbol{K}_{1l}\Delta t(\boldsymbol{u}_l^{P+1} + \boldsymbol{u}_l^P) + \\
&\quad 2(\alpha\boldsymbol{M}_{1l} + \beta\boldsymbol{K}_{1l})(\boldsymbol{u}_l^{P+1} - \boldsymbol{u}_l^P)] \\
\ddot{\boldsymbol{u}}_1^{P+1} &= -\ddot{\boldsymbol{u}}_1^P + \frac{2}{\Delta t}(\dot{\boldsymbol{u}}_1^{P+1} - \dot{\boldsymbol{u}}_1^P)
\end{aligned}\right\} \tag{21}$$

The relations given by Eq. (21) are the step-by-step integration relations for computing the response quantities about each node in the finite element discrete model. For any node $i$, the response quantities can be computed by Eq. (21), only if the node number '1' in Eq. (21) is replaced by 'i'.

## 4 Transmitting artificial boundary in time domain

The explicit formulas for computing the response quantities of the nodes within the artificial boundary (not including the nodes on the artificial boundary) in the finite element discrete model are presented above. However, in order to bring about the step-by-step computation for the nodal response quantities, it is necessary to have the step-by-step formulas for computing the response quantities of nodes on the artificial boundaries. It is just the problem to be solved by any artificial boundary method.

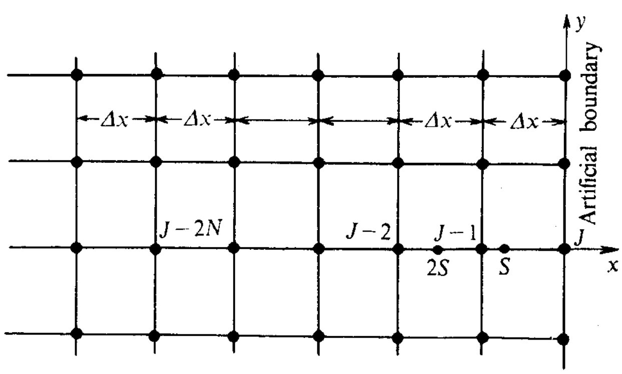



Figure 4 Transmitting Boundary in the discrete model in space

The artificial boundary is a kind of fictitious boundaries set up in the original continuous medium when the infinite continuous medium is considered by finite medium model. Because the original medium is continuous in essence on the artificial boundaries, it is necessary to ensure that the wave transmission characters on the artificial boundary are consistent with those of the wave in the original continuous mediums, that is, the wave is not reflected but entirely transmitted on the artificial boundaries. Multi-transmitting Boundary (Extrapolation Boundary) is a kind of artificial boundary methods which directly simulate the wave transmission characters on artificial boundaries. In this artificial boundary method, the wave field is divided into the inward propagating wave and the outward propagating wave, and Multi-transmitting Boundary mainly simulates the outward propagating wave (simulates the motions of the mediums caused by the outward propagating wave on artificial boundaries). The approximate explicit recurrence relationship of the motions between the mediums on artificial boundaries and those within artificial boundaries can be obtained by Multi-transmitting Boundary in time domain.

If assuming that $u_{IJ}$ and $u_{RJ}$ are the displacement quantities caused by the inward propagating wave and outward propagating wave at the node J on artificial boundaries, we have

$$u_J^{P+1} = u_{IJ}^{P+1} + u_{RJ}^{P+1} \tag{22}$$

where, $u_{IJ}^{P+1}$ is the value of the incident wave field at the point J on the artificial boundary at the time $P+1$.

For the artificial boundary in the finite element discrete model shown in Figure 4, following relationship is given by Generalized Transmitting Boundary (Li, 1993; Liao and Li, 1994)

$$u_{RJ}^{P+1} = \sum_{n=1}^{N}(-1)^{n+1}C_n^N u_{n,J}^{P+1} + \sum_{l=1}^{M}(-1)^{l+1}C_l^M(u_{RJ-l}^{P+1} - \sum_{n=1}^{N}(-1)^{n+1}C_n^N u_{n,J-l}^{P+1}) \tag{23}$$

in which,

$$u_{K,J'}^{P'} = \frac{1}{2}(1-S)(2-S)u_{K-1,J'}^{P'-1} + S(2-S)u_{K-1,J'-1}^{P'-1} + \frac{1}{2}S(S-1)u_{K-1,J'-2}^{P'-1} \tag{24}$$

and

$$u_{0,J'}^{P'} = u_{RJ'}^{P'} \tag{25}$$

where,

$$P' = P+1,\ P,\ \cdots,\ P+1-N; \qquad K = N,\ N-1,\ \cdots,\ 1;$$
$$J' = J,\ J-1,\ \cdots,\ J-2N-M$$

In above equations, $M$ and $N$ are the parameters controlling the simulating accuracy of Generalized Transmitting Boundary, in general case, $M=1$, $N=1$ or $M=1$, $N=2$; $S=C_a\Delta t/\Delta x$, $C_a$ is artificial wave velocity. The response velocity quantity $\dot{u}$ is also given by Eq. 23, only the u in Eq. 23 is replaced by $\dot{u}$.

## 5 Examples and analyzing results

The basic formulas about the explicit finite element-finite difference method for analyzing the response of two-dimensional local site are given above, and according to above theory and relative formulas, a computer program is compiled. As an example for the application of our method, the topographic effects and viscous damping effects of mediums in local site on earthquake ground motion are computed and brief analyses about the computing results are done.

After artificial boundaries are set up in the local topographic site model and the finite element discretization of the site model in space is done, the site model is as shown in Figure 5. The values of S wave velocity and P wave velocity of the site medium are 1 400 m/s and 2 000 m/s, the density of the site medium is 2 000 kg/m$^3$. In computation, we assume that the earthquake incident wave is plane propagating wave SH wave, SV wave or P wave at 30° incident angle.

The incident wave time-histories and computing response displacement time-histories at the observation points shown in Figure 5 about some analyzing cases are shown in Figure 6 and Figure 7. Our computing response results show that the effects of the viscous damping are that the higher frequency components in the response motion on the free surface of site (comparing with

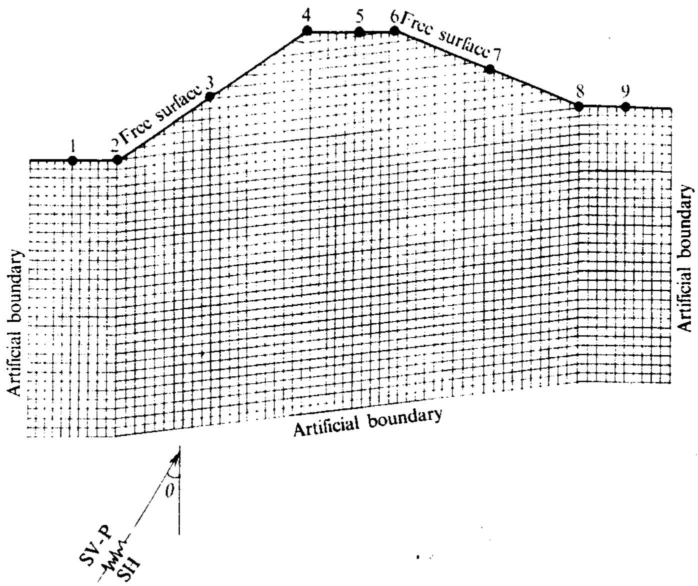
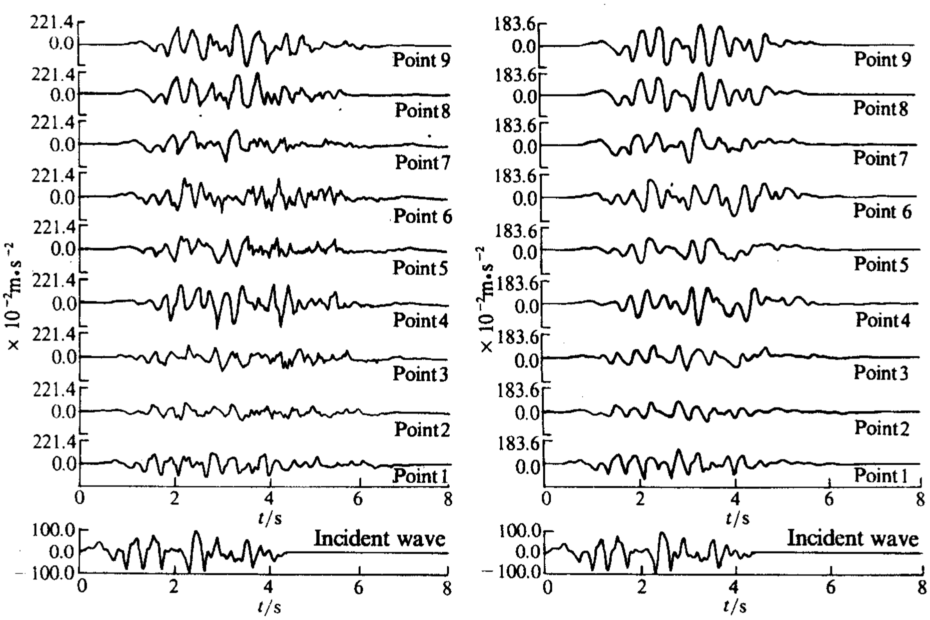


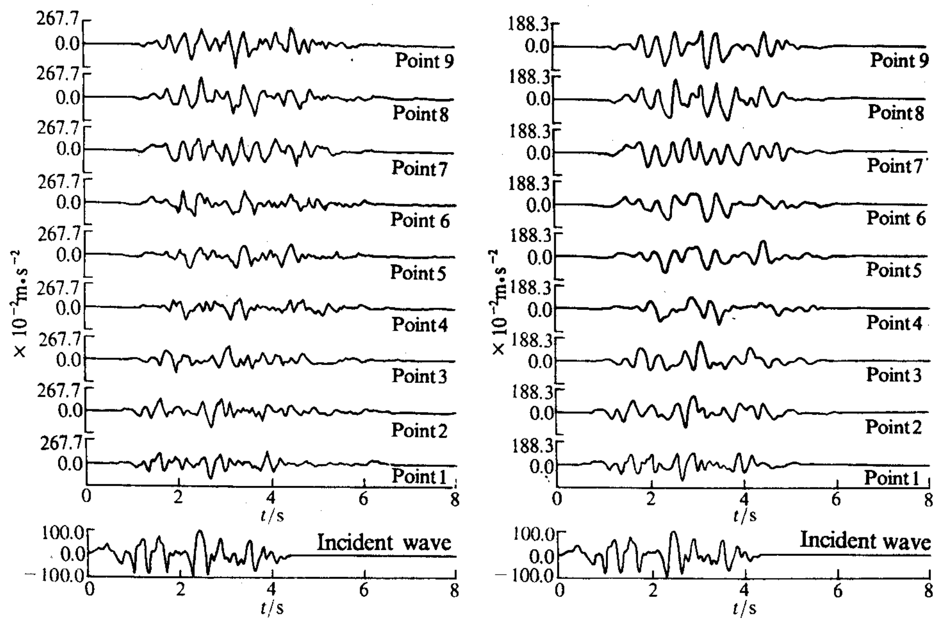

Figure 5 Finite element model of visco-elastic topographic site




SV-P wave problem SV wave incident Horizontal response SV-P wave problem SV wave incident Horizontal response
Damping parameters $\alpha=0.0$, $\beta=0.0$ Displacement time-history Damping parameters $\alpha=0.0$, $\beta=0.0$ Displacement time-history

Figure 6 Computing results of the effects of visco-elastic topography on earthquake ground motion

the incident wave) are obviously decreased, and with increase of the viscous damping value, the degree of the effects increases. These computing results may be explained by the following two




SV-P wave problem SV wave incident Vertical response SV-P wave problem SV wave incident Vertical response
Damping parameters $\alpha=0.0$, $\beta=0.0$ Displacement time-history Damping parameters $\alpha=0.0$, $\beta=0.0$ Displacement time-history

Figure 7 Computing results of the effects of visco-elastic topography on earthquake ground motion

points: 1) because of the existence of the viscous damping, the effects of the energy dissipation of wave motion appear, and the effects become more serious with increase of the damping value, on the other hand, because the damping effects of medium are considered by the Rayleigh damping theory (in which $\alpha=0, \beta\neq 0$), the degree of the energy dissipation increases with increase of the frequency value of the wave propagating in medium, that is to say that there are strong filtering effects to the high frequency waves; 2) the explicit finite difference method proposed by the author is used to solve the dynamic equations for the visco-elastic system in our method, and the finite difference method possesses the numerical calculation property of strong high frequency dissipation and the dissipation effects sharply increase with increase of the damping value, therefore, with increase of the damping value, it is certain that much higher frequency components in the response motions decrease greatly.

The computing results also show that much higher frequency components (comparing with the components in the incident wave) appear in the computing response motions in the case of elastic medium. The reason is that explicit finite difference method does not possess the property of high frequency dissipation in the case of no viscous damping. In this case, the numerical integration method can not control the accumulation of the high frequency errors which include the errors caused by both the discretization of time and space and the high frequency instability property of "transmitting boundary" in the calculating processes.

## 6 Conclusions

The paper has introduced the basic idea about the high accuracy explicit finite element-finite difference method for analyzing the earthquake response of the two-dimensional local site, and has given the detail computing formulas about the problem of visco-elastic topography, some examples have been shown about the analyses of the effects of topography on earthquake ground motion. The basic theory of our method is also suitable for the problem of three-dimensional local site, including the problem of inhomogeneous and nonlinear local site. For the nonlinear problem, the explicit method presented in this paper can give full play to its superiority, that is, comparing with other methods, the computing time and computer memory space needded by the computing program of our method decrease on a large scale. This is just the main purpose why we put forward the high accuracy integration formulas to perfect the explicit finite element-finite difference method in this paper.

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